Music and Mathematics: Diagrammatic Reasoning in the 14th Century

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Abstract. The 14th century diagrams in the focus of this paper deal with geometric sequences, fractional powers and recursion, which are mathematical topics relevant also for music theory: defining consonant intervals, comparing the sizes of musical intervals and dividing them into equal parts, constructing musical scales with several equal steps. These diagrams might have paved the way to mathematical concepts further developed only in the 16th and 17th centuries.

Keywords: Geometric Sequences, Musical Intervals, Irrational Ratios

1 Complete graphs and medieval ratio theory

Arc diagrams can be traced back to Boethius (c. 477ï 524). In the following, complete arc diagrams with four nodes are called *tetraktys-diagrams*. The nodes in the elementary Pythagorean tetraktys, Fig. 1a, represent the natural numbers 1 to 4 and their ratios, the *perfect consonances* of the Pythagoreans: the double octave (4 : 1), the twelfth (3 : 1), the octave (2 : 1 = 4 : 2), the fifth (3 : 2) and the fourth (4 : 3). The symmetrical partition of the octave into two fourths and a whole tone (9 : 8) of Fig. 1b is a common Pythagorean tetraktys-diagram for the proportion 6 : 8 : 9 : 12.

Nicole Oresme makes a novel use of the tetraktys-diagram. He scales the classical tetraktys, Fig. 1a, proportionally down into a single octave by halving all its constituent intervals and obtains the configuration shown in Fig. 1i with the proportion $1 : \sqrt{2} : \sqrt{3} : 2$ containing square roots. Another diagram from the same Oresme manuscript showing a complete graph with seven nodes is transcribed and interpreted in Fig. 1k. [1] Instead of numbers the nodes are regular polygons, inscribed and circumscribed to circles of the same radius. The analysis of the original arc labels shows that the nodes are to interpret as areas of the polygons. These areas form musical intervals ï if bisecting the Pythagorean intervals is accepted as a musical operation. The equal bisection of (musical) ratios involving irrational numbers is beyond the rigid Pythagorean number universe. In the same manuscript, Oresme discusses and illustrates the trisection of the *Diapason cum Diapente*, the twelfth (3 : 1), which is equivalent to ñtripling the cubeò: finding the sides of a cube with the threefold volume or $\sqrt[3]{3}$. Possibly, Michael Stifel in the the 16th century took inspiration from Oresme, when he developed the concept of fractional (and negative) powers. [2]

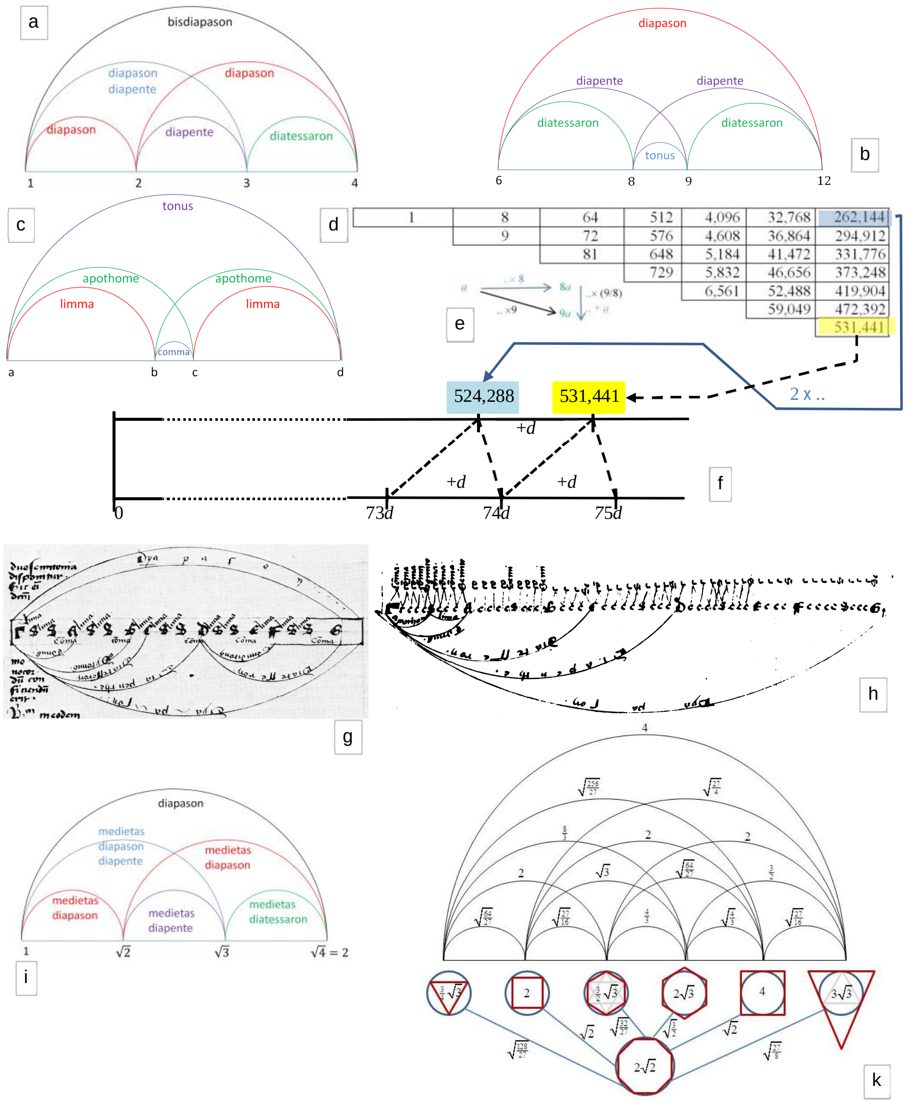

Fig. 1. Musical diagrams occurring in 14th century manuscripts.

2 Commas and superparticular ratios

The symmetric division of the tone (9 : 8) according to Jacobus Leodiensis [3], see Fig. 1c, into two *limmae* (256 : 243) and a *comma* (531,441 : 524,228) has the proportion 497,664 : 524,228 : 531,441 : 559,872. It explains the relations between the smallest common musical intervals. The comma is also defined as the interval difference between six tones and an octave as illustrated in Fig. 1d to 1f. If the powers of 8 are calculated first, the other numbers in the triangular matrix can be determined with simple additions as indicated in Fig. 1e (by the author), a procedure that can be applied to arbitrary geometrical sequences with a superparticular common ratio. Medieval Boethius manuscripts include tables for 3/2, 5/4 and sometimes also for 9/8. [4]

Boethius knew that the ratio of the Pythagorean comma is between 75/74 and 74/73 and that the limma measures more than three but little less than four commas. Since he had no logarithms, he and his followers used superparticular ratios $(n + 1)/n$ in order to measure arbitrary musical intervals smaller than the octave. The method is visualized in a diagram by Walter Odington [5], see Fig. 1f. The ticks on the two imagined parallel rulers have a common distance of d = 7,153 units ï the difference between the terms of the Pythagorean comma.

3 Enharmonic and microtonal «monochords»

The octave of the Pythagorean diatonic scale has five whole tone steps (9 : 8) and two limmae (256 : 243). In the first ñmonochordò diagram by Leodiensis, Fig. 1g, each of the five tones is divided into two limmae and a comma generating an enharmonic scale with 17 steps per octave. And in Fig. 1h Boethiusôs size comparison of the comma to the limma is applied to each of the five limmae of the enharmonic scale from Fig. 1g leading to a microtonal scale with 53 steps per octave: 41 Pythagorean commas and 12 slightly deficient commas. [6] It can also be obtained by piling 52 just fifths (3:2) modulo octave. In the 1660ies Nicolaus Mercator and Isaac Newton discussed the division of the octave into 53 *equal* steps in order to obtain a system of pitches that optimally approaches Pythagorean and just intonation scales. [7]

4 Conclusion

The mathematicians of the 14th century had sufficient means to successfully and elegantly tackle sophisticated music theoretical problems. The little known diagrams by Nicole Oresme, Walter Odington, and Jacobus Leodiensis provide important puzzle stones and missing links to the history of music as a mathematical discipline.

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